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The special relativistic shock tube

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The shock-tube problem has served as a popular test for numerical hydrodynamics codes. The development of relativistic hydrodynamics codes has created a need for a similar test problem in relativistic hydrodynamics. The analytical solution to the special relativistic shock-tube problem is presented here. The relativistic shock-jump conditions and rarefaction solution which make up the shock tube are derived. The Newtonian limit of the calculations is given throughout.

1. Introduction

The shock-tube problem has long served as a popular test for numerical hydrodynamics codes, as in Sod (1978), because the solution in the perfect-gas case can be obtained analytically, and because the shock tube is a time-dependent problem whose solution contains discontinuities. Discontinuities pose a challenge for numerical solution methods, since finite-difference approximations to derivatives only make sense when applied to continuous functions.

The more recent development of relativistic hydrodynamics codes by Centrella & Wilson (1984) and Thompson (1985) has created a need for analytical solutions to relativistic fluid-dynamics problems, against which the codes may be tested. This paper develops the analytical solution to the special relativistic shock-tube problem for the constant-γ equation of state, superceding the more limited solution obtained by Wilson and the author in Centrella & Wilson (1984).

2. Description of the problem

The fluid is characterized by a rest mass density ρ ($= mn$, where n is the number of particles per volume and m is the particle rest mass, measured in the fluid frame), a pressure p, and a 4 velocity U ($= \mathrm{d}x/\mathrm{d}\tau$, the derivative of the fluid-element coordinate with respect to proper time), assuming motion is allowed only in the x-direction. We also have the following auxiliary relations:

$$V = U/w, \tag{1}$$

$$w = (1+U^2)^{\frac{1}{2}} = 1/(1-V^2)^{\frac{1}{2}}, \tag{2}$$

$$p = (\gamma-1)\,\epsilon, \tag{3}$$

$$\sigma = \rho+\epsilon+p = \rho+\frac{\gamma}{\gamma-1}p, \tag{4}$$

$$s = p\rho^{-\gamma} \quad \text{(provided } \gamma = \text{constant)}, \tag{5}$$

where V is the physical velocity ($\mathrm{d}x/\mathrm{d}t$, where t is the coordinate time), w is the Lorentz factor, ϵ is the thermal energy density, and s is a measure of the entropy. The units are such that the speed of light is unity (hence $V < 1$).

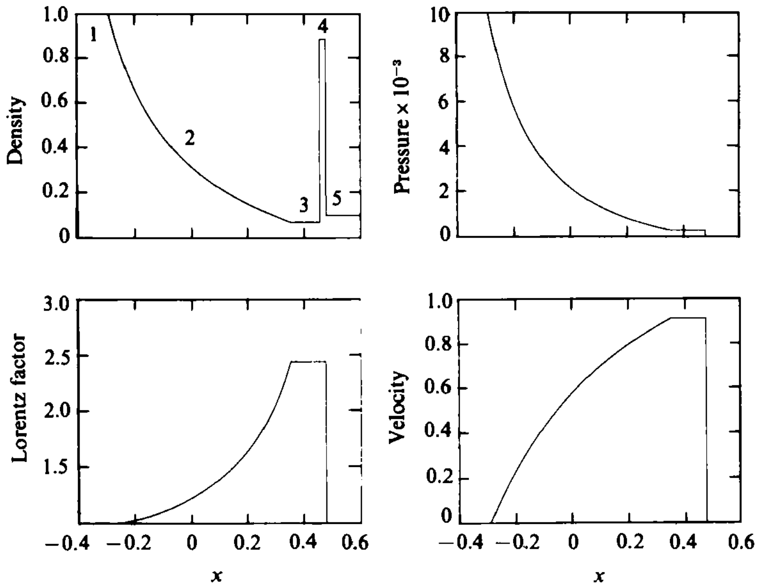

FIGURE 1. A special relativistic shock tube.

It remains to specify an equation of state for the fluid, by defining the quantity γ which appears in (3). For a monatomic Newtonian gas $\gamma = \frac{5}{3}$. For a monatomic relativistic gas, γ is a function of temperature, starting at $\frac{5}{3}$ in the low-temperature limit where $p \ll \rho$, and decreasing monotonically to the high-temperature limit of $\frac{4}{3}$ when $p \gg \rho$. The complete temperature dependence is derived by Lightman *et al.* (1975), and given by Thompson (1985) as

$$\gamma = \left[1+\beta\left(\frac{K_1(\beta)}{K_2(\beta)}-1\right)+3\right]^{-1}, \quad \beta = \frac{\rho}{p}, \tag{6}$$

where K_n is the modified Bessel function of the second kind of order n. Solving the rarefaction and shock equations with γ given by (6) is beyond the scope of this paper, and would probably require a numerical integration of the differential equations. The case γ = constant is considered below. The assumption of constant γ is valid in the Newtonian ($\gamma = \frac{5}{3}$) and extreme relativistic ($\gamma = \frac{4}{3}$) limits, and should be a good approximation for the case $p \approx \rho$ provided that the ratio p/ρ does not vary much in the problem at hand.

The shock-tube problem is as follows. At time $t = 0$ we have a stationary fluid in two different states, separated by a partition. Each state is uniform. The partition is removed at $t = 0$, and the higher-pressure fluid expands, pushing the lower-pressure fluid in front of it. For definiteness, assume that the initial interface is at $x = 0$, and that the higher-pressure gas is on the left ($x < 0$). Then as time goes on, the initial jump becomes a similarity solution in $\xi = x/t$, comprised of five regions numbered 1–5 from left to right, whose general appearance is given in figure 1.

Region 1 is the undisturbed leftmost state, whose right boundary at $x = \xi_1 t$ moves to the left with time ($\xi_1 < 0$). It is stationary ($V_1 = 0$) and characterized by values $\rho = \rho_1$ and $p = p_1$ from the initial conditions on the left.

Region 2 is a rarefaction. Its leftmost boundary (the rarefaction wave) moves to the left at constant velocity $\xi_1 < 0$. The fluid in the rarefaction moves to the right with a velocity $V = V_2(\xi)$. The density and pressure are $\rho = \rho_2(\xi)$ and $p = p_2(\xi)$. The density and pressure decrease with ξ, while the velocity increases. The right boundary is at $x = \xi_2 t$.

Region 3 is a plateau, with $\rho = \rho_3$, $p = p_3$, $V = V_3$ all constant. Its left boundary is at $x = \xi_2 t$, and its right boundary is a contact discontinuity at $x = \xi_3 t$.

Region 4 is a second plateau, with $\rho = \rho_4$, $p = p_4$, $V = V_4$ all constant. Its left boundary is the contact discontinuity at $x = \xi_3 t$, and its right boundary is a right-moving shock wave at $x = \xi_4 t$.

Region 5 is the undisturbed region to the right of the shock at $x = \xi_4 t$, into which the shock is moving. Its constant state of $\rho = \rho_5$, $p = p_5$, $V = 0$ is the same as the initial conditions on the right.

The figure shows a shock tube defined by $\rho_1 = 1$, $p_1 = 10^4$, $\rho_5 = 0.1$, $p_5 = 10$, and $\gamma = \frac{4}{3}$ throughout (the extreme relativistic limit, consistent with the specified pressure and density), at time $t = 0.5$. Note that the density jump across the shock is 8.9, exceeding the Newtonian maximum of 7.

3. The shock jump

The two principal elements of the solution are the rarefaction and the shock wave. The jump conditions for a relativistic shock wave are derived in this section.

A shock wave is a surface of discontinuity, across which the density, pressure and normal velocity are discontinuous. A reference system can always be found in which the shock is stationary, and fluid flows through the standing shock. Since the fluid quantities are finite (though discontinuous) at the shock, it follows that the mass, momentum and energy fluxes are continuous across the shock in this frame (otherwise the fluid would 'pile up' at the shock).

The shock-jump conditions relate the states on either side of the shock, and are based on the continuity of fluxes in the shock's comoving frame, as demonstrated by Taub (1948, 1967). To derive these conditions, we begin by writing the one-dimensional relativistic fluid equations of Thompson (1985), which represent conservation of mass, momentum and energy respectively:

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial x}(\rho U) = 0 \rightarrow \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho V) = 0, \tag{7}$$

$$\frac{\partial}{\partial t}(\sigma w U)+\frac{\partial}{\partial x}(\sigma U^2+p) = 0 \rightarrow \frac{\partial}{\partial t}(\rho V)+\frac{\partial}{\partial x}(\rho V^2+p) = 0, \tag{8}$$

$$\frac{\partial}{\partial t}(\sigma w^2-p)+\frac{\partial}{\partial x}(\sigma w U) = 0 \rightarrow \frac{\partial}{\partial t}(\epsilon+\tfrac{1}{2}\rho V^2)+\frac{\partial}{\partial x}[(\epsilon+p+\tfrac{1}{2}\rho V^2)V] = 0, \tag{9}$$

where the arrows represent the Newtonian limits. (Strictly speaking, the right-hand side of (9) is not the Newtonian limit of the left-hand side, since the left-hand side represents the conservation of total energy, including rest mass, while the right-hand side represents conservation of thermal and kinetic energies only. However, one performs essentially the same calculations with either side to get the shock-jump conditions below.)

Now consider a coordinate system $x't'$ in which the shock is at rest, moving with a constant 4 velocity U_s (and a physical velocity $V_s = U_s/w_s < 1$) with respect to the original system. The Lorentz transformation connecting the two systems is

$$x' = -U_s t+w_s x \rightarrow -V_s t+x, \tag{10}$$

$$t' = w_s t-U_s x \rightarrow t. \tag{11}$$

Thus the derivatives transform as

$$\frac{\partial}{\partial t} = w_s \frac{\partial}{\partial t'} - U_s \frac{\partial}{\partial x'} \rightarrow \frac{\partial}{\partial t'} - V_s \frac{\partial}{\partial x'}, \tag{12}$$

$$\frac{\partial}{\partial x} = -U_s \frac{\partial}{\partial t'} + w_s \frac{\partial}{\partial x'} \rightarrow \frac{\partial}{\partial x'}, \tag{13}$$

and the equation

$$\frac{\partial f}{\partial t} + \frac{\partial g}{\partial x} = 0 \tag{14}$$

becomes

$$\frac{\partial}{\partial t'}(w_s f - U_s g) + \frac{\partial}{\partial x'}(w_s g - U_s f) = 0 \rightarrow \frac{\partial f}{\partial t'} + \frac{\partial}{\partial x'}(g - V_s f) = 0 \tag{15}$$

in the $x't'$ system. (If U_s varies with time or the geometry is not rectangular the derivation becomes more complicated, but the results obtained below are unchanged.)

Let the two sides of the shock be denoted 1 and 2, and the shock position by x'_s. Then integrating (15) from $x'_s - h$ to $x'_s + h$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t'} \int_{x'_s-h}^{x'_s+h} (w_s f - U_s g)\,\mathrm{d}x + (w_s g - U_s f)\Big|_{x'_s-h}^{x'_s+h} = 0. \tag{16}$$

As $h \rightarrow 0$ the time-derivative term vanishes. The result is

$$w_s g_1 - U_s f_1 = w_s g_2 - U_s f_2 - U_s f_2 \rightarrow g_1 - V_s f_1 = g_2 - V_s f_2. \tag{17}$$

Replacing f and g by the densities and fluxes appearing in (7)–(9) gives

$$\begin{aligned} \rho_1(w_s U_1 - U_s w_1) &= \rho_2(w_s U_2 - U_s w_2) \\ &\rightarrow \rho_1(V_1 - V_s) = \rho_2(V_2 - V_s), \end{aligned} \tag{18}$$

$$\begin{aligned} w_s(\sigma_1 U_1^2 + p_1) - U_s \sigma_1 w_1 U_1 &= w_s(\sigma_2 U_2^2 + p_2) - U_s \sigma_2 w_2 U_2 \\ &\rightarrow \rho_1 V_1^2 + p_1 - \rho_1 V_1 V_s = \rho_2 V_2^2 + p_2 - \rho_2 V_2 V_s, \end{aligned} \tag{19}$$

$$\begin{aligned} w_s \sigma_1 w_1 U_1 - U_s(\sigma_1 w_1^2 - p_1) &= w_s \sigma_2 w_2 U_2 - U_s(\sigma_2 w_2^2 - p_2) \\ &\rightarrow \left(\tfrac{1}{2}\rho_1 V_1^2 + \frac{\gamma}{\gamma-1} p_1\right) V_1 - \left(\tfrac{1}{2}\rho_1 V_1^2 + \frac{1}{\gamma-1} p_1\right) V_s \\ &= \left(\tfrac{1}{2}\rho_2 V_2^2 + \frac{\gamma}{\gamma-1} p_2\right) V_2 - \left(\tfrac{1}{2}\rho_2 V_2^2 + \frac{1}{\gamma-1} p_2\right) V_s, \end{aligned} \tag{20}$$

assuming that γ is the same on both sides of the shock.

The above equations are not very useful as written. It is more convenient to rewrite them in other forms. For the Newtonian limit, we have the following results:

$$\frac{\rho_1}{\rho_2} = 1 - \frac{2}{\gamma+1}\left[1 - \frac{\gamma p_1}{\rho_1(V_1 - V_s)^2}\right], \tag{21}$$

$$\frac{V_2 - V_s}{V_1 - V_s} = \frac{\rho_1}{\rho_2}, \tag{22}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left[\frac{\rho_1(V_1 - V_s)^2}{\gamma p_1} - 1\right], \tag{23}$$

$$\frac{\rho_1}{\rho_2} = \frac{\gamma + 1 + (\gamma - 1) p_2/p_1}{\gamma - 1 + (\gamma + 1) p_2/p_1}, \tag{24}$$

$$V_s = V_1 \pm \left(\frac{\rho_2}{\rho_1}\frac{p_2-p_1}{\rho_2-\rho_1}\right)^{\frac{1}{2}} = V_2 \pm \left(\frac{\rho_1}{\rho_2}\frac{p_1-p_2}{\rho_1-\rho_2}\right)^{\frac{1}{2}}, \tag{25}$$

$$V_s = \frac{\rho_2 V_2 - \rho_1 V_1}{\rho_2-\rho_1}. \tag{26}$$

Although p_2/p_1 can take on any value from 0 to ∞, (24) implies that

$$\frac{\gamma-1}{\gamma+1} \leqslant \frac{\rho_1}{\rho_2} \leqslant \frac{\gamma+1}{\gamma-1}, \tag{27}$$

i.e. the maximum density jump is finite and depends on γ. The density of the gas can change at most by a factor of 4 if $\gamma = \frac{5}{3}$, or 7 if $\gamma = \frac{4}{3}$.

The relativistic shock equations are much more complicated and do not lend themselves to expressions that are both simple and general. It is more fruitful to look at two special cases instead, the strong shock and general shock cases, both with $U_1 = V_1 = 0$, $w_1 = 1$. We get

$$\rho_1 V_s = \rho_2 w_2 (V_s - V_2), \tag{28}$$

$$\sigma_2 U_2^2 + p_2 = V_s \sigma_2 w_2 U_2 + p_1, \tag{29}$$

$$V_s(\sigma_2 w_2^2 - p_2) - \sigma_2 w_2 U_2 = V_s(\sigma_1 - p_1), \tag{30}$$

from which we obtain the relations

$$\frac{\rho_2}{\rho_1} = \frac{p_2 - p_1 + \frac{\gamma}{\gamma-1} p_2 U_2^2}{w_2(p_2-p_1) - \rho_1 U_2^2}, \tag{31}$$

$$V_s = \frac{p_2 - p_1 + \frac{\gamma}{\gamma-1} p_2 U_2^2}{\left(\rho_1 + \frac{\gamma}{\gamma-1} w_2 p_2\right) U_2}, \tag{32}$$

$$V_s\left(\sigma_2 w_2^2 - p_2 - \rho_1 - \frac{1}{\gamma-1} p_1\right) - \sigma_2 w_2 U_2 = 0. \tag{33}$$

If the shock is strong, p_1 can be neglected throughout, giving

$$p_2 = \rho_1[\gamma w_2^2 - (\gamma-1) w_2 - 1], \tag{34}$$

$$\rho_2 = \rho_1 \frac{\gamma w_2 + 1}{\gamma-1} = \rho_1\left[\frac{\gamma+1}{\gamma-1} + \frac{\gamma}{\gamma-1}(w_2-1)\right], \tag{35}$$

$$V_s = \frac{1 + \frac{\gamma}{\gamma-1} U_2^2}{\rho_1 + \frac{\gamma}{\gamma-1} w_2 p_2} \frac{p_2}{U_2}. \tag{36}$$

Thus if the pre-shock density ρ_1 and the post-shock velocity U_2 are known, the remaining variables are given. Note that the density jump can be arbitrarily large, unlike the Newtonian case.

In the more general case $p_1 \neq 0$, and simple analytic solutions for p_2, ρ_2 and V_s as functions of ρ_1, p_1 and U_2 are not available. However, if ρ_1, p_1 and U_2 are given, then ρ_2 and V_s can be considered functions of the unknown variable p_2 through (31) and (32), and (33) then defines the only valid p_2 which satisfies all three equations. Write

$$f(p_2) = V_s\left(\sigma_2 w_2^2 - p_2 - \rho_1 - \frac{1}{\gamma-1}p_1\right) - \sigma_2 w_2 U_2. \tag{37}$$

Then the equation

$$f(p_2) = 0 \tag{38}$$

can be solved by a Newton's-method iterative technique, as described in the Appendix.

4. The rarefaction

The rarefaction solution in region 2 is a continuous function, and is obtained by solving the fluid equations. The solution technique is similar to that of Mathews (1971), who found a similarity solution for the hydromagnetic free expansion of a relativistic gas.

The density and velocity equations for constant γ, given by Thompson (1985), can be written as

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho U) = 0 \rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V) = 0, \tag{39}$$

$$\sigma\left(\omega\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}\right) - \gamma p U\left(\frac{\partial w}{\partial t} + \frac{\partial U}{\partial x}\right) + \frac{\partial p}{\partial x} = 0 \rightarrow \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{40}$$

where the Newtonian limits are given by the arrows. The entropy is constant and equal to that of the original left state, the flow is adiabatic, and γ is assumed constant, so that

$$p = s\rho^{\gamma}, \tag{41}$$

where $s = p_1 \rho_1^{-\gamma}$ is constant. Assuming that all quantities depend on x and t only through the combination $\xi = x/t$ gives

$$\frac{\partial}{\partial t} = -\frac{\xi}{t}\frac{\mathrm{d}}{\mathrm{d}\xi}, \tag{42}$$

$$\frac{\partial}{\partial x} = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}\xi}, \tag{43}$$

and we get, after a little algebra,

$$(V-\xi)\frac{\mathrm{d}\rho}{\mathrm{d}\xi} + \frac{\rho}{w}(1 - V\xi)\frac{\mathrm{d}U}{\mathrm{d}\xi} = 0 \rightarrow (V-\xi)\frac{\mathrm{d}\rho}{\mathrm{d}\xi} + \rho\frac{\mathrm{d}V}{\mathrm{d}\xi} = 0, \tag{44}$$

$$\frac{c^2}{w\rho}\frac{\mathrm{d}\rho}{\mathrm{d}\xi} + [V - \xi - c^2 V(1 - V\xi)]\frac{\mathrm{d}U}{\mathrm{d}\xi} = 0 \rightarrow \frac{c^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}\xi} + (V-\xi)\frac{\mathrm{d}V}{\mathrm{d}\xi} = 0, \tag{45}$$

where the speed of sound c is given by

$$c^2 = \frac{\gamma p}{\sigma} \rightarrow \frac{\gamma p}{\rho}. \tag{46}$$

Equations (44) and (45) are homogeneous equations for $\mathrm{d}\rho/\mathrm{d}\xi$ and $\mathrm{d}U/\mathrm{d}\xi$ (or $\mathrm{d}V/\mathrm{d}\xi$), and possess non-trivial solutions only if the determinant of the coefficient matrix vanishes. Thus

$$w^2(V-\xi)^2 - c^2(1 - V\xi)[1 + w^2 V(V-\xi)] = 0 \rightarrow (V-\xi)^2 - c^2 = 0. \tag{47}$$

We see immediately that the point where $V = 0$ has $\xi = \xi_1 = -c_1$ (choosing the minus sign to correspond with the orientation chosen), where c_1 is the sound speed of the undisturbed fluid in region 1. That is, the rarefaction wave moves into region 1 at the local speed of sound. If the fluid were expanding into a vacuum on the other side (i.e. if region 5 were a vacuum), the leading edge would have expanded infinitely and cooled to the point where $c = 0$, in which case $\xi = V \equiv V_{max}$. (Note that ξ goes from $-c_1$ to V_{max} as V goes from 0 to V_{max}.) However, the value of V_{max} cannot be determined from (47).

Since (47) holds, we know that (44) and (45) are equivalent. Thus we can write

$$\frac{d\rho}{dU} = -w\frac{\rho}{c^2}\frac{V-\xi}{1+w^2V(V-\xi)} \rightarrow \frac{d\rho}{dV} = -\frac{\rho}{c^2}(V-\xi). \tag{48}$$

Solving (47) for ξ gives

$$\xi = V - \frac{c}{w^2(1-cV)} \rightarrow \xi = V - c. \tag{49}$$

Substituting for ξ in (48) gives

$$\frac{d\rho}{dU} = -\frac{\rho}{c(1+U^2)^{\frac{1}{2}}} \rightarrow \frac{d\rho}{dV} = -\frac{\rho}{c}. \tag{50}$$

Now define

$$y = \left(\frac{\gamma s}{\gamma-1}\right)^{\frac{1}{2}} \rho^{(\gamma-1)/2} = \left(\frac{\gamma}{\gamma-1}\frac{p}{\rho}\right)^{\frac{1}{2}} = y_1\left(\frac{\rho}{\rho_1}\right)^{(\gamma-1)/2}. \tag{51}$$

Separating (50) and integrating from $y_1 = y(\rho_1)$ to $y_2 = y(\rho_2)$ with respect to y, and integrating from $U = 0$ to U_2 with respect to U gives

$$\rho_2 = \rho_1\left(\frac{y_2}{y_1}\right)^{2/(\gamma-1)}, \tag{52}$$

$$p_2 = p_1\left(\frac{\rho_2}{\rho_1}\right)^{\gamma}, \tag{53}$$

$$y_2 = \frac{f^2(y_1) - f^{\sqrt{(\gamma-1)}}(U_2)}{2f(y_1)f^{\frac{1}{2}\sqrt{(\gamma-1)}}(U_2)}, \tag{54}$$

where

$$f(x) \equiv x + (1+x^2)^{\frac{1}{2}}, \tag{55}$$

$$x = \frac{f^2(x)-1}{2f(x)}. \tag{56}$$

Now ρ and p are known functions of U. Note that ρ decreases as U increases, as required by (50). We can also solve for U as a function of ρ (through y as defined in (51)):

$$U_2 = \frac{f^{4/\sqrt{(\gamma-1)}}(y_1) - f^{4/\sqrt{(\gamma-1)}}(y_2)}{2[f(y_1)f(y_2)]^{2/\sqrt{(\gamma-1)}}}. \tag{57}$$

Given U and ρ, V is defined by

$$V_2 = \frac{U_2}{(1+U_2^2)^{\frac{1}{2}}}, \tag{58}$$

and ξ is given by (49). Note that

$$c_2^2 = \frac{(\gamma-1)y_2^2}{1+y_2^2}. \tag{59}$$

The Newtonian limit is simpler. We have the relations

$$\rho_2 = \rho_1\left(1-\frac{\gamma-1}{2}\frac{V_2}{c_1}\right)^{2/(\gamma-1)}, \tag{60}$$

$$V_2 = \frac{2c_1}{\gamma-1}\left[1-\left(\frac{\rho_2}{\rho_1}\right)^{(\gamma-1)/2}\right], \tag{61}$$

$$\rho_2 = \rho_1\left[\frac{\gamma-1}{\gamma+1}\left(\frac{2}{\gamma-1}-\frac{\xi}{c_1}\right)\right]^{2/(\gamma-1)}, \tag{62}$$

$$V_2 = \frac{2}{\gamma+1}(\xi+c_1), \tag{63}$$

$$c_2 = c_1\frac{\gamma-1}{\gamma+1}\left(\frac{2}{\gamma-1}-\frac{\xi}{c_1}\right). \tag{64}$$

5. The complete solution

Now for the remainder of the shock tube. We have already seen that $\xi_1 = -c_1$. We also know that $\rho_3 = \rho_2(\xi_2)$, $p_3 = p_2(\xi_2)$, $V_3 = V_2(\xi_2)$, $p_4 = p_2(\xi_2)$, $V_4 = V_2(\xi_2)$. Since the contact discontinuity moves at the local fluid velocity, it follows that $\xi_3 = V_3$. Thus we need to find ξ_2, ξ_4 and ρ_4.

We already have the rarefaction solution, which can be put in the form $p = p_r(U)$; i.e. the pressure in the rarefaction is a given function of U once the original left state is specified. The rarefaction relation between p and U also holds in regions 3 and 4 (p and U are simply constant with position there, but are related to each other in the same way as in the rarefaction). On the other hand, the shock-jump conditions of §3 (given values for ρ_5 and p_5) provide a second expression $p = p_s(U)$ for the pressure behind a shock, when the shocked fluid has a velocity U. Thus we have two equations in two unknowns, which give U_4 (and hence p_4) as the solution to

$$F(U_4) \equiv p_r(U_4) - p_s(U_4) = 0. \tag{65}$$

The function $F(U)$ defined in (65) has the value $p_1 - p_5 > 0$ for $U = 0$ and decreases as U increases, passing through zero at $U = U_4$. A Newton's-method iteration of the form

$$U^{n+1} = U^n - \frac{F(U^n)}{\mathrm{d}F/\mathrm{d}U|_{U^n}} \tag{66}$$

gives a sequence of approximations which converge to U_4. For an initial guess U^0, guaranteed to converge to U_4, choose the velocity for adiabatic expansion into a vacuum ($U_{\max}$) given by (57) with $\rho_2 = y_2 = 0$:

$$U_{\max} = \tfrac{1}{2}[f^{2/\surd(\gamma-1)}(y_1) - f^{-2/\surd(\gamma-1)}(y_1)]. \tag{67}$$

A numerical approximation to the derivative of the form

$$\frac{\mathrm{d}F}{\mathrm{d}U} = \frac{F(U+h)-F(U-h)}{2h}$$

is used, with $h = \min(U^0/100, |U^n - U^{n-1}|/10)$.

Having obtained U_4, we get p_4 from the shock-jump conditions or the rarefaction equations. Then V_4, V_3, ξ_3, p_3, ρ_3, and ξ_2 follow from the continuity conditions across the contact discontinuity and the rarefaction solution. We get ρ_4 and the shock velocity ξ_4 from the shock-jump conditions, and the solution is complete.

A complication occurs if we want to evaluate the solution at a particular x and t, corresponding to a particular $\xi = x/t$. If ξ is not in region 2 the evaluation is straightforward, since it is one of the constant states. If ξ is in region 2, there is a problem, because ρ and p are given in terms of U, not ξ, and we need to determine U (or, equivalently, V) as a function of ξ. Equation (49) is a nonlinear equation for V, given ξ. We solve

$$f(V) \equiv V - \xi - \frac{c}{w^2(1-cV)} = 0 \tag{68}$$

for V, where w and c are known functions of V, and where V is known to lie in the interval $0 \leqslant V \leqslant V_{\max} = U_{\max}/w_{\max} \leqslant 1$. Once more Newton's method is used, just as in (66). In the Newtonian limit $\xi(V)$ is linear, so we take as a starting guess the linear interpolation

$$V^0 = -V_{\max}\frac{f(0)}{f(V_{\max})-f(0)}. \tag{69}$$

In the Newtonian case more of the work can be done analytically. Matching the rarefaction and shock pressures ultimately gives a nonlinear equation for the shock velocity $V_s (= \xi_4)$. The equation is

$$V_s - \frac{c_5^2}{V_s} = \frac{\gamma+1}{\gamma-1}c_1\left\{1-\left[\frac{2}{\gamma+1}\frac{\rho_5}{p_1}\left(V_s^2 - \frac{\gamma-1}{2\gamma}c_5^2\right)\right]^{(\gamma-1)/2\gamma}\right\}. \tag{70}$$

Denoting the right-hand side by A, we can write

$$V_s^2 - AV_s - c_5^2 = 0, \tag{71}$$

which can be solved by the rather simple iterative algorithm

$$V_s^{n+1} = \tfrac{1}{2}\{A^n + [(A^n)^2 + 4c_5^2]^{\frac{1}{2}}\}, \tag{72}$$

$$A^n = A(\overline{V}_s^n), \tag{73}$$

$$\overline{V}_s^n = \tfrac{1}{2}(\overline{V}_s^{n-1} + V_s^n), \tag{74}$$

where

$$\overline{V}_s^0 = c_5, \tag{75}$$

and c_5 is the sound speed in region 5. In the vacuum limit, $\rho_5 = c_5 = 0$, and $V_s \rightarrow (\gamma+1/\gamma-1)\,c_1$.

Given V_s, the values of p_4 and ρ_4 are obtained from the shock-jump equations. The other variables follow from the rarefaction solution and the continuity conditions for the contact discontinuity. Since the Newtonian rarefaction is written explicitly in terms of ξ, no difficulties are encountered in evaluating it for a given ξ-value.

6. Summary

This paper presents an analytical solution to the relativistic shock-tube problem, subject to the constraint γ = constant. The solution is 'analytical' in the sense that it does not require the numerical solution of the original differential equations, although the numerical solution of algebraic equations is required, largely because the most general shock jump is considered. This solution therefore supercedes that given in Centrella & Wilson (1984) in two respects: the earlier work entailed the numerical solution of the differential equations; and considered only the more tractable case of strong shock jumps. The current solution is not only easier to evaluate but covers a broader class of problems.

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Appendix. Solution of the relativistic shock-jump equations

We want to solve for p_2 in $f(p_2) = 0$, with $f(p_2)$ given by (37). Given an initial guess p_2^0 for p_2, a sequence of improved estimates is obtained from

$$p_2^{n+1} = p_2^n - \frac{f(p_2^n)}{\mathrm{d}f/\mathrm{d}p_2|_{p_2^n}}, \tag{A 1}$$

where

$$\frac{\mathrm{d}f}{\mathrm{d}p_2} = \frac{\partial f}{\partial p_2} + \frac{\partial f}{\partial V_s}\frac{\mathrm{d}V_s}{\mathrm{d}p_2} + \frac{\partial f}{\partial \rho_2}\frac{\mathrm{d}\rho_2}{\mathrm{d}p_2}, \tag{A 2}$$

and where

$$\frac{\partial f}{\partial p_2} = \frac{\gamma}{\gamma-1} w_2(w_2 V_s - U_2) - V_s, \tag{A 3}$$

$$\frac{\partial f}{\partial V_s} = \sigma_2 w_2^2 - p_2 - \rho_1 - \frac{1}{\gamma-1} p_1, \tag{A 4}$$

$$\frac{\partial f}{\partial \rho_2} = w_2(w_2 V_s - U_2), \tag{A 5}$$

$$\frac{\mathrm{d}V_s}{\mathrm{d}p_2} = \frac{1}{B}\left[1 - \frac{\gamma}{\gamma-1} U_2(w_2 V_s - U_2)\right], \tag{A 6}$$

$$\frac{\mathrm{d}\rho_2}{\mathrm{d}p_2} = \frac{1}{C}\left[\rho_1\left(1 + \frac{\gamma}{\gamma-1} U_2^2\right) - \rho_2 w_2\right], \tag{A 7}$$

$$V_s = \frac{A}{B}, \tag{A 8}$$

$$\rho_2 = \frac{A}{C}\rho_1, \tag{A 9}$$

$$A = p_2 - \rho_1 + \frac{\gamma}{\gamma-1} p_2 U_2^2, \tag{A 10}$$

$$B = \left(\rho_1 + \frac{\gamma}{\gamma-1} w_2 p_2\right) U_2, \tag{A 11}$$

$$C = w_2(p_2 - p_1) - \rho_1 U_2^2. \tag{A 12}$$

The function $f(p_2)$ is a monotonically decreasing function for $p_2 > p_{\min}$, where

$$p_{\min} = p_1 + \frac{\rho_1 U_2^2}{w_2} \tag{A 13}$$

is the value of p_2 for which C is zero and f is infinite. A reasonable starting guess is then $p_2^0 = 2p_{\min}$. To prevent a new estimate p_2^{n+1} from going below $p_{\min}$, perform the replacement

$$p_2^{n+1} \to \max[p_2^{n+1}, (1+10^{-10})\, p_{\min}] \tag{A 14}$$

after each iteration. On a computer with 64-bit arithmetic (double precision for 32-bit machines, single precision for the Cray 1 and Cray X-MP machines) this algorithm converges to the correct answer provided that $U_2 < 700$. For $U_2 > 700$, rounding errors make the results invalid. (The convergence properties were determined by comparison to a double precision implementation of the algorithm on the Cray X-MP.)

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